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A NEW METHOD OF FINDING DIFFERENTIALS—A SUGGESTION FOR MATHEMATICIANS.

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It seems to me that much of the obscurity and uncertainty now attending the first stages in the study of the Calculus can be avoided by means of the following suggestions:

The CALCULUS is a method of finding values by means of *rates of variation.*

A DIFFERENTIAL COEFFICIENT is any number by which it is necessary to multiply the rate at which one variable changes in order that it may be equal to the rate of variation of another.

Now suppose we have a variable as x, which increases *uniformly* at the rate of one unit of the measure of increase in one unit of the measure of time; we write the differential dx. On this supposition x becomes consecutively 1, 2, 3, 4, 5, &c.—and $dx = 1$. Now suppose we have $dy^2 = 2$. We have another series 1, 3, 5, 7, &c., or possibly 0, 2, 4, 6, &c.

Suppose we have x and y and $u = x + y$; what is the differential of u, the function of $x + y$, or the sum of two variables?

Suppose $dx = 1$, as in the former case, and $dy = 2$, then we have denoting the successive moments by the letters a, b, c, etc.,

	a,	b,	c,	d.				
$x =$	1,	2,	3,	4,	5,	etc., successively, and		$dx = 1$.
$y =$	1,	3,	5,	7,	9,	etc.,	"	$dy = 2$.
Sums,	2,	5,	8,	11,	14,		"	

Here, we see at once that the differential of the sum, $x + y$, is 3, or the sum of the differentials 1 and 2, or $du = dx + dy$. Or if we subtract one series from the other, term for term, we find in the same way that the differential of the difference of two variables, is the difference of their differentials.

Or if two horses of unequal strength, one of which can raise a weight at the rate of one mile an hour, and the other at the rate of two miles an hour, both pulling together they will raise it at the rate of three miles an hour. Or if they pull in opposite directions, they will raise it one mile an hour in the direction of the effort of the strongest horse; we have, therefore, $d(x + y) = dx + dy$ and $d(x - y) = dx - dy$.

Now let us try the product of two variables, x and y; we have $u = xy$. As before, let a, b, c, etc., denote different moments of time, and we have moments,

		a,	b,	c,	d.	
$x =$	1,	2,	3,	4,	5,	etc., successively, and $dx = 1$.
$y =$	1,	3,	5,	7,	9,	" " $dy = 2$.
Products $=$	1,	6,	15,	28,	45,	"
Differences $=$	5,	9,	13,	17.		

If we take the values of one variable, as x at the beginning and at the end of any moment, as d, for example, add them together and divide the sum by two, we get the average of increase for that moment. If now we treat the values of the other variable for that moment in the same way, and then multiply the average of each variable by the differential of the others and add the sums, we get the same result as before, thus:

$$4 + 5 = 9 \text{ divided by } 2 = 4\tfrac{1}{2}.$$
$$7 + 9 = 16 \text{ " by } 2 = 8.$$
$$\text{Now, } 4\tfrac{1}{2} \times 2 = 9 \text{ and } 8 \times 1 = 8.$$

And $8 + 9 = 17$, which is the increment of the product of the variables for that moment, or the rate is 17 divided by 1.

Nor is it at all necessary that the variables should increase at any uniform rate, the result will be the same whatever may be the rate or law of increase of the several factors.

If, in the cases before us, $x = y$ and $dx = dy$, then $xdy + ydx$ becomes $2xdx$, which is the differential expression for x^2. Or if we have instead, three variables, x, v and z, equal to each other, and their differentials are also respectively equal, we have the differential of a cube, or $d(x^2) = 3x^2dx$.

After having found the differential of the product of two variables, it is a much simpler and easier way to find that of three or more algebraically. Thus, for xyz let $s = xy$, then $sz = xyz$, and differentiating sz by the formula for the produst of two variables, we have $sdz + zds$. Substituting for s its value, xy, and for ds its value, $xdy + ydx$, and we shall have the result, $xydz + zydx + zxdy$.

And in fact with the two already given, namely, the sum and difference and the product of two variables, we can find all the others of any kind or order *algebraically* as in the above example.

I will however give one more example of reference to series for the sake of another fact that it will develop.

Suppose we have the fraction $x \div y$ with the desire to find its differential. Let x equal 1, 2, 3, &c., successively, and y equal 1, 3, 5, 7, &c., then $dx = 1$ and $dy = 2$. Considering them in separate moments as above we have, moment a, b, c, d,

$$\frac{1}{1},\quad \frac{2}{3},\quad \frac{3}{5},\quad \frac{4}{7}, \text{ \&c., \&c.}$$

Take for example the moment b, and the fraction increases, or rather *decreases* trom $\frac{10}{15}$ to $\frac{9}{15}$, or $\frac{1}{15}$. If we take the next moment or c it decreases from $\frac{21}{35}$ to $\frac{20}{35}$ or $\frac{1}{35}$. For the next moment the decrease is $\frac{1}{63}$.

Here we observe that the numerator of these amounts of decrease for each successive moment is uniform,– 1. If now we take the fraction for any moment as $2 \div 3$ and multiply the differential of the numerator 1 by the denominator for the moment 3, and from their product $= 3$ subtract the product of the numerator by the differential of the denominator $2 \times 2 = 4$, we have -1, which is the constant numerator found above.

Now whenever both the numerator and the denominator of the fraction increase or decrease uniformly—that is, each of them have any constant for their differential coefficient, we should have the same phenomenon, namely, a constant numerator for the amount of the change and this constant is always the differential of the numerator minus the differential of the denominator; that is, when the rate of change is uniform in both the members of the fraction, the usual form $ydx - xdy$ is the same as $dx - dy$, and the coefficients y and x are unnecessary.

If now we turn to consider the danominators obtained above for the successive moments, we shall see that we obtained them by multiplying the value at the beginning of a moment as 3 into the value at the end of that moment as 5—giving us 15, 35, 63, &c. If we are to regard the $-1 \div 15$ as the *amount* of change—this difference is important. But if we are to regard the moment as an indivisible amount of time, and the $-1 \div 15$ as the rate of change *at* that moment rather than the amount of change *during* any moment or amount of time, the difference between y' and y'' becomes nothing and $y' \times y''$ is the same as y^2. Hence the formula $(ydx - xdy) \div y^2$, and when x and y taken separately change uniformly, we have $(dx - dy) \div y^2$ for the differential of the fraction $x \div y$.

And in all this there is no assumption that the differential is small, or that it may be treated as nothing on one side of an equation while we use it as something on the other; and there is no need of any such assumption. For the sake of convenience the unit or moment of time should be made the same far all the variables that enter into any one equation. But it may be different for each set of variables between which we may have occasion to make an equation.

The method of finding differentials above suggested is based on the principle that "the *amount* of change *during* any moment, considered as a quantity of time, is, when divided by the unit of time, the *rate* of change *at* the moment, considered as an indivisible point of time." Hence we first find the amount of change in a moment in numbers and replace those numbers, factor by factor, with the algebraic symbols representing the variables and the differentials of each of them. We have then a general expression for the rate of change of the compound variable.

It may be said that this is after all only experimental. It is merely a numerical computation, and has not the generality and abstraction of form that mathematicians demand.

Perhaps so. But, then, there is no algebraic method of proving these formulae, that has yet fallen under my notice, that does not encounter some one or more of the difficulties I mentioned at the beginning of this paper. There is no one of them that I know of that does not involve an absurdity, and that may not be shown to be false by what is known as the indirect method of proof as *reductio ad absurdum.*

But again. We must accept some things that cannot be proved algebraically or by general formula. The multiplication table cannot be *proved* in that way, that I know of. I do not think that any man has ever yet proved algebraically, or by the use of letters, that $3 \times 4 = 12$. He may say, "let a represent 3 and b 4, and then the product will be ab." So indeed we may call it, and so we may write it. But who shall prove to us that ab is twelve rather than thirteen or eleven?

There are doubtless cases in which we must make the unit of increase exceedingly small; thus, in the rectification of curves, we can measure the chord of an arc, the arc itself we cannot measure; therefore, if we will express the rate of variation of the arc in units of increment, which are exceedingly small, we may take the length of its chord, which we can measure, for the length of an arc, which we cannot, and we get an approximate value for the circumference of the circle in the terms of its radius. But it seems to me we had better leave the explanation of these cases until they actually occur in the course of the application of the Calculus.

The necessity for so regarding the differential is not found in the nature of the differential at all,—but it is found in the nature of the cases to which we apply the Calculus.

I take the more interest in this matter because of the great importance I attach to the Calculus as a means of mental discipline and culture; and this value does not arise at all from its being a difficult study —one that requires patience, perseverence and concentration of thought. It is rather because it puts the mind into a new attitude in regard to all things, and enables the man of thought to see them in a new light and in new relations. I can hardly regard any one as capable of comprehending the highest, the most general and the most comprehensive truths, without the power and the habit of looking at them from the point of view to whice the study of the Calculus will, of necessity, carry him.

But I believe that the subject has been very unnecessarily involved in metaphysical subtleties—not to say absurdities—and that the difficulty of comprehending it, and the consequent disinclination to study it so common in all our schools and colleges, has arisen chiefly from this unnecessary embarrassment. Adopt the explanation I have now given, treat the differential dx as you treat any other factor, and the Calculus may be made as intelligible as the multiplication table. It may be applied to the simplest operations in arithmetic, or in the proof and solution of the simplest problem in geometry—may, in fact, be understood by all persons. and be none the less powerful and wonderful as a means of science in the attainment of its most recondite facts and laws, on that account.

It will also be observed that the formulae given by my method do not differ at all from those in common use. The only difference is in the explanation given to them and the method of finding and proving them. Nor will this method, so far as I can see, while making them intelligible and bringing them within the easy comprehension of all persons, put the learner in any position of disadvantage in reference to the higher and more difficult questions that he must encounter in his further pursuit of knowledge or in the practical applications of it to the various purposes of life.

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ON THE DIFFERENTIAL EQUATIONS OF DYNAMICS.

BY G. W. HILL.

The general formula of dynamics is

$$\Sigma\left[\left(m\frac{d^2x}{dt^2}-X\right)\delta x+\left(m\frac{d^2y}{dt^2}-Y\right)\delta y+\left(m\frac{d^2z}{dt^2}-Z\right)\delta z\right]=0.$$

In the usual treatment of this equation, we have been asked to attribute to the symbols δx, δy, δz, &c., the signification they have in the calculus of variations. This however is unnecessary, except when we wish to deduce from it the principle of least action; and the student, unacquainted with this calculus, may regard these symbols as multipliers, which, when all the points of the system are free, have any finite values we please, but when the coordinates are restricted to satisfy an equation $U = 0$, are subject to the condition

$$\frac{dU}{dx}\delta x+\frac{dU}{dy}\delta y+\frac{dU}{dz}\delta z+\text{\&c.}=0,$$

an equation which, for brevity, we shall write $\delta U = 0$.

We shall confine our attention to those cases in which the equations of condition and the accelerating forces are functions of the coordinates and the time only, and in which the latter are equivalent to the partial differential coefficients of a single function Ω taken with respect to the coordinates whose acceleration they express.

Whenever a function as U involves, in addition to x, y, z, &c., their first differential coefficients with respect to the time, quantities which we shall denote by x', y', z', &c., we shall suppose that δU involves, besides the terms written above, the following